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# THE BIFURCATION OF THE EQUILIBRIA OF MECHANICAL SYSTEMS WITH SYMMETRICAL POTENTIAL†

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The problem of the bifurcation of the equilibrium positions of conservative systems whose potential energy is independent of the signs of the variables that occur in it is discussed. A method for the sequential determination of the equilibrium positions of such systems, beginning with the trivial ones, in order of their increasing complexity, is proposed. © 1998 Elsevier Science Ltd. All rights reserved.

It is well known that the problem of finding the equilibrium positions or steady motions of conservative mechanical systems and of investigating their stability reduces to the problem of analysing the critical points of the potential energy or reduced potential energy of this system. Critical points of the potential energy are found from the system of non-linear algebraic equations, the complete investigation of which often involves considerable computational difficulties. Fortunately, in mechanical problems the potential energy is often invariant to some change of variables, which is due to the presence of some discrete groups of symmetries in these problems. This property enables one to determine the simplest classes of solutions of this system of algebraic equations solely from considerations of symmetry. The presence or absence of other (asymmetrical) classes of solutions remains an open question. The solution of this problem depends largely on the properties of the simplest solutions when the physical parameters of the mechanical system change. If the index of the second variation of the potential energy, calculated for some simplest solution, changes when the physical parameters change, then, by bifurcation theory [1], other classes of solutions certainly exist. The method of constructing these solutions, it is true, still remains an open question.

In this paper we describe a method for the sequential determination of the non-trivial equilibrium positions (steady motions) in order of their increasing complexity. This method is based on the symmetry properties of a mechanical system and the properties of the second variation of the potential energy, calculated for the equilibrium position (steady motion) of the previous level of complexity, beginning with the trivial one.

1. Suppose $V(x; p)$ is the potential energy of the system (initial or reduced), $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the vector of the system coordinates (in the general case, dependent), and $\mathbf{p} = (p_1, \ldots, p_m)^T$ is the vector of the physical parameters (the superscript $T$ denotes transposition).

We will assume that

$$V((-1)^{\mathbf{s}}\mathbf{x};\ \mathbf{p}) \equiv V(\mathbf{x};\ \mathbf{p}) \quad (\mathbf{s} = (s_1, \ldots, s_n)^T, \ \ s_i = 0 \ \text{or} \ 1)$$

$$(-1)^{\mathbf{s}}\mathbf{x} = ((-1)^{s_1}x_1, \ldots, (-1)^{s_n}x_n)^T$$

In other words, we will assume that the function $V(\mathbf{x}; \mathbf{p})$ can be represented in the form

$$V(\mathbf{x}; \mathbf{p}) = W(\boldsymbol{\xi}; \mathbf{p}) \quad (\xi_i = x_i^2, \ i = 1, \ldots, n) \tag{1.1}$$

In many applied problems of mechanics it is usually more convenient to use dependent variables. We will therefore assume that the $x$ coordinates are connected by the relation $f(\mathbf{x}) = 0$, which defines a compact space of the system configurations, which is also invariant under the replacement $\mathbf{x} \to (-1)^{\mathbf{s}}\mathbf{x}$.

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For simplicity we will confine ourselves initially to the case of an $(n-1)$-dimensional sphere

$$f(\mathbf{x}) \equiv x_1^2+\ldots+x_n^2-1=0 \tag{1.2}$$

To find the critical points of the function (1.1) in the set (1.2) we will introduce the function $2F = W + \lambda f$, where $\lambda$ is a Lagrange undetermined multiplier, and we will write the conditions for it to be steady with respect to the variables $x$, $\lambda$

$$\frac{\partial F}{\partial x_i}=\left(\frac{\partial W}{\partial \xi_i}+\lambda\right)x_i=0, \quad i=1, \ldots, n; \quad \frac{\partial F}{\partial \lambda}=\frac{1}{2}f=0 \tag{1.3}$$

It is obvious that system (1.3) has the trivial solutions

$$x_\alpha=\pm 1, \quad x_j=0 \quad (j\neq\alpha), \quad \alpha=1, \ldots, n \tag{1.4}$$

$$\lambda=\lambda_\alpha=-\left[\frac{\partial W}{\partial \xi_\alpha}\right]_{(1.4)} \tag{1.5}$$

To determine the nature of the critical points (1.4) we will calculate the second variation $\delta^2 F$ of the function $F$ in the linear manifold $\delta f = 0$ (taking (1.5) into account), which, for solution (1.4) has the form $\delta x_\alpha = 0$

$$2\delta^2 F=\sum_{j\neq\alpha} c_j^{(\alpha)}(\delta x_j)^2; \quad c_j^{(\alpha)}=\left[\frac{\partial W}{\partial \xi_j}-\frac{\partial W}{\partial \xi_\alpha}\right]_{(1.4)}=c_j^{(\alpha)}(\mathbf{p}) \tag{1.6}$$

Thus the following assertion holds.

*Assertion* 1.1. The function (1.1) on the sphere (1.2) always takes critical values at the points (1.4), and its second variation at these points always has the form of the sum of squares (1.6).

We will consider some of the trivial solutions (1.4) and we will assume that, for certain values of the parameters, one of the Poincaré coefficients of the second variation (1.6) changes sign, while the remaining coefficients keep their sign. Then, for these values of the parameters, the index of the second variation changes and, by bifurcation theory, other solutions branch off from the trivial solution considered. In the general case, a search for these non-trivial solutions is a difficult problem to solve. However, in this case (a symmetric potential and a symmetric configuration space) we can suggest a quite simple method of constructing these solutions.

Suppose for solution (1.4) the coefficient $c_\beta^{(\alpha)}$ changes sign in a certain set

$$\mathbf{P}_\beta^\alpha=\{\mathbf{p}\in\mathbf{R}^m : c_\beta^{(\alpha)}(\mathbf{p})=0\} \tag{1.7}$$

where $c_j^{(\alpha)} \neq 0$ when $j \neq \beta$ and $\mathbf{p} \in \mathbf{P}_\beta^{(\alpha)}$. We will seek solutions of system (1.3) in the form ($\varphi \neq 0/\text{mod } \pi/2$)

$$x_\alpha=\pm\cos\varphi, \quad x_\beta=\pm\sin\varphi, \quad x_j=0 \quad (j\neq\alpha, \beta) \tag{1.8}$$

Substituting relations (1.8) into (1.3), we obtain

$$\frac{\partial W_{\alpha\beta}}{\partial \xi_\alpha}+\lambda=0, \quad \frac{\partial W_{\alpha\beta}}{\partial \xi_\beta}+\lambda=0 \tag{1.9}$$

where $W_{\alpha\beta} = W_{(1.8)}$, while the remaining equations of system (1.3) are satisfied identically with respect to $\varphi$ and $\lambda$. Subtracting the first equation of system (1.9) from the second, we have

$$\Phi_{\alpha\beta}(\varphi;\ \mathbf{p})\equiv\frac{\partial W_{\alpha\beta}}{\partial \xi_\beta}-\frac{\partial W_{\alpha\beta}}{\partial \xi_\alpha}=0 \tag{1.10}$$

According to the above assumptions, the function $\Phi_{\alpha\beta}(0; \mathbf{P}) \equiv c_\beta^{(\alpha)}(\mathbf{p})$ changes sign when $\mathbf{p} \in \mathbf{P}_\beta^\alpha$. We will fix the value of $p_+$, close to the set $\mathbf{P}_\beta^\alpha$, so that $\Phi_{\alpha\beta}(0; \mathbf{P}) > 0$. Here, by the continuity for all $\varphi \in (-\delta, \delta)$, where $\delta > 0$ is sufficiently small, we have

$$\Phi_{\alpha\beta}(\varphi;\ \mathbf{p}_+) > 0 \tag{1.11}$$

Similarly it can be shown that when $\varphi \in (-\delta, \delta)$

$$\Phi_{\alpha\beta}(\varphi;\ \mathbf{p}_-) < 0 \tag{1.12}$$

(the value of $p_-$ is close to the set $\mathbf{P}_\beta^\alpha$ and $\Phi(0; \mathbf{p}_-) < 0$). It follows from relations (1.11) and (1.12) (taking (1.1) into account), that Eq. (1.10) has the pair of solutions

$$\varphi = \pm\varphi_{\alpha\beta}(\mathbf{p}) \tag{1.13}$$

where $\Phi_{\alpha\beta}(0; \mathbf{P}) = 0$ for $\mathbf{p} \in \mathbf{P}_\beta^\alpha$; here (see (1.9))

$$\lambda = \lambda_{\alpha\beta} = -\left[\frac{\partial W}{\partial \xi_\alpha}\right]_{(1.8)} = -\left[\frac{\partial W}{\partial \xi_\beta}\right]_{(1.8)} \tag{1.14}$$

Note that for values of $\mathbf{p}$ close to $\mathbf{P}_\beta^{(\alpha)}$, system (1.3) has no solutions, unlike (1.4) and (1.8), since it follows from (1.3) that $x_j = 0$ ($j \neq \alpha, \beta$; $\mathbf{p}$ is close to $\mathbf{P}_\beta^\alpha$) in view of the assumption $c_j^{(\alpha)}(\mathbf{p}) \neq 0$ ($j \neq \beta$; $\mathbf{p} \in \mathbf{P}_\beta^{(\alpha)}$). The solutions (1.8) certainly exist for values of the parameter $p$ which lie in the neighbourhood of the set $\mathbf{P}_\beta^\alpha$, but they may also exist outside this neighbourhood (the latter depends on the properties of the function $\Phi_{\alpha\beta}$ in each specific case).

To determine the nature of the critical points (1.8) (see also (1.13) and (1.14)) we will calculate the second variation $\delta^2 F$ in a linear manifold $\delta f = 0$, which, for the solution (1.8), has the form

$$(\cos\varphi_{\alpha\beta})\delta x_\alpha + (\sin\varphi_{\alpha\beta})\delta x_\beta = 0$$

(by virtue of the symmetry of the problem here and henceforth all the solutions (1.4), (1.8), etc. are taken with the upper signs).

Hence, $\delta x_\alpha = -(\operatorname{tg}\varphi_{\alpha\beta})\delta x_\beta$ and

$$2\delta^2 F = \sum_{j\neq\alpha} c_j^{(\alpha\beta)}(\delta x_j)^2 \tag{1.15}$$

$$c_j^{(\alpha\beta)} = \left[\frac{\partial W}{\partial \xi_j} - \frac{\partial W}{\partial \xi_\alpha}\right]_{(1.8)} = c_j^{(\alpha\beta)}(\mathbf{p})\ (j \neq \beta)$$

$$c_\beta^{\alpha\beta} = \left[\frac{\delta^2 W}{\delta\xi_\alpha^2} - 2\frac{\delta^2 W}{\delta\xi_\alpha\delta\xi_\beta} + \frac{\delta^2 W}{\delta\xi_\beta^2}\right]_{(1.8)} \sin^2\varphi_{\alpha\beta} = c_\beta^{(\alpha\beta)}(\mathbf{p})$$

Thus the following assertion holds.

*Assertion* 1.2. If bifurcation values of the parameters (1.7) exist for the trivial solution (1.4), the function (1.1) on the sphere (1.2) takes critical values at points of the form (1.8), and its second variation at these points has the form of the sum of the squares (1.15).

The coefficient $c_\beta^{(\alpha\beta)}(p)$ obviously vanishes, but it does not change sign when $\mathbf{p} \in \mathbf{P}_\beta^{(\alpha)}$, since in this case $\varphi_{\alpha\beta}(\mathbf{p})$ vanishes. If this coefficient vanishes and changes sign for certain other values of the parameters (due to the coefficient of $\sin^2\varphi_{\alpha\beta}$), then it can be shown in the same way as before that the function $\varphi_{\alpha\beta}(\mathbf{p})$ loses its uniqueness. In this case, there is one other class of solutions of the form (1.8) (but with a different function $\varphi_{\alpha\beta}(\mathbf{p})$ and not in the neighbourhood of the set ($\varphi = 0$; $\mathbf{P}_\beta^{(\alpha)}$) in the space $\mathbf{S} \times \mathbf{R}^m$($\varphi \in \mathbf{S}$, $\mathbf{p} \in \mathbf{R}^m$)).

The case in which one of the other Poincaré coefficients, say $c_\gamma^{(\alpha\beta)}$ ($\gamma \neq \beta$), changes sign in a certain set

$$\mathbf{P}_\gamma^{(\alpha\beta)} = \{\mathbf{p} \in \mathbf{R}^m : c_\gamma^{(\alpha\beta)}(\mathbf{p}) = 0\} \tag{1.16}$$

is of considerable interest. Here, as previously, we will assume that the remaining coefficients of the second variation (1.15) do not vanish when $\mathbf{p} \in \mathbf{P}_\gamma^{\alpha\beta}$. Then, the following solutions branch off from solution (1.18) ($\varphi, \psi \neq 0/\mathrm{mod}\ \pi/2$)

$$x_\alpha = \pm\cos\varphi\cos\psi, \quad x_\beta = \pm\sin\varphi\cos\psi$$
$$x_\gamma = \pm\sin\psi, \quad x_j = 0\ (j \neq \alpha,\ \beta,\ \gamma) \tag{1.17}$$

In fact, substituting (1.17) into (1.3) we obtain

$$\frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\alpha} + \lambda = 0, \quad \frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\beta} + \lambda = 0, \quad \frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\gamma} + \lambda = 0 \tag{1.18}$$

where $W_{\alpha\beta\gamma} = W_{(1.17)}$, and the remaining equations of system (1.3) are satisfied identically with respect to $\varphi$, $\psi$ and $\lambda$. Subtracting the first equation of system (1.18) from the third and second, we have

$$\mathbf{\Phi}_{\alpha\beta\gamma}(\varphi;\psi;\mathbf{p}) \equiv \left[\frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\gamma} - \frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\alpha}\right] = 0 \tag{1.19}$$

$$\mathbf{\Psi}_{\alpha\beta\gamma}(\varphi;\psi;\mathbf{p}) \equiv \left[\frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\beta} - \frac{\partial W_{\alpha\beta\gamma}}{\partial\xi_\alpha}\right] = 0 \tag{1.20}$$

Note that when $\psi = 0$, Eq. (1.20) becomes (1.10), while the left-hand side of Eq. (1.19) becomes the coefficient $c_\gamma^{(\alpha\beta)}$ of the quadratic form (1.15). Consequently, we can determine $\varphi = \bar{\varphi}_{\alpha\beta\gamma}(\psi, \mathbf{p})$ from Eq. (1.20) for all $\psi$ close to zero, where $\bar{\varphi}_{\alpha\beta\gamma}(0, \mathbf{p}) = \varphi_{\alpha\beta}(\mathbf{p})$. Substituting $\bar{\varphi}_{\alpha\beta\gamma}$ into (1.19) we have

$$\overline{\mathbf{\Phi}}_{\alpha\beta\gamma}(\psi,\ \mathbf{p}) \equiv \mathbf{\Phi}_{\alpha\beta\gamma}(\bar{\varphi}_{\alpha\beta\gamma}(\psi,\ \mathbf{p});\psi;\mathbf{p}) = 0 \tag{1.21}$$

Existence of the solution $\psi = \pm\psi_{\alpha\beta\gamma}(\mathbf{p})$ of Eq. (1.21) is proved in the same way as the proof that the solution of Eq. (1.10) exists. Hence, system (1.19), (1.20) has a solution of the form

$$\varphi = \pm\varphi_{\alpha\beta\gamma}(\mathbf{p}), \quad \psi = \pm\psi_{\alpha\beta\gamma}(\mathbf{p}) \tag{1.22}$$

$$(\varphi_{\alpha\beta\gamma}(\mathbf{p}) = \bar{\varphi}_{\alpha\beta\gamma}(\pm\psi_{\alpha\beta\gamma}(\mathbf{p});\ \mathbf{p}))$$
$$\lambda = \lambda_{\alpha\beta\gamma} = -\left[\frac{\partial W}{\partial\xi_\rho}\right]_{(1.17)} \quad (\rho = \alpha,\ \beta,\ \gamma) \tag{1.23}$$

Note that for values of $\mathbf{p}$ close to $\mathbf{P}_\gamma^{(\alpha\beta)}$, system (1.3) has no solutions, unlike (1.4), (1.8) and (1.17), since it follows from (1.3) that $x_j = 0$ ($j \neq \alpha, \beta, \gamma$; $\mathbf{p}$ is close to $\mathbf{P}_\gamma^{(\alpha\beta)}$). The solutions (1.17) certainly exist for values of the parameters $\mathbf{p}$ which lie in the neighbourhood of the set $\mathbf{P}_\gamma^{(\alpha\beta)}$, but they may also exist outside this neighbourhood.

To determine the nature of the critical points (1.17) we must calculate (taking (1.22) and (1.23) into account) the second variation $\delta^2 F$ in the linear manifold $\delta f = 0$, which for the solution (1.17) has the form

$$(\cos\varphi_{\alpha\beta\gamma}\cos\psi_{\alpha\beta\gamma})\delta x_\alpha + (\sin\varphi_{\alpha\beta\gamma}\cos\psi_{\alpha\beta\gamma})\delta x_\beta + (\sin\psi_{\alpha\beta\gamma})\delta x_\gamma = 0$$

Hence

$$2\delta^2 F = [2\delta^2 F]^{(\beta\gamma)} + [2\delta^2 F]^{(j)} \quad (j \neq \alpha,\ \beta,\ \gamma) \tag{1.24}$$

$$[2\delta^2 F]^{(\beta\gamma)} = a_{\beta\beta}(\delta x_\beta)^2 + 2a_{\beta\gamma}(\delta x_\beta)(\delta x_\gamma) + a_{\gamma\gamma}(\delta x_\gamma)^2 \tag{1.25}$$

$$[2\delta^2 F]^{(j)} = \sum_{j \neq \alpha,\beta,\gamma} c_j^{(\alpha\beta\gamma)}(\delta x_j)^2 \tag{1.26}$$

Explicit expressions for the coefficients $a_{\rho\sigma}$ ($\rho, \sigma = \beta, \gamma$) and $c_j^{(\alpha\beta\gamma)}$ ($j \neq \alpha, \beta, \gamma$) are fairly lengthy and are therefore not given here.

Thus the following assertion holds.

*Assertion* 1.3. If bifurcation values of the parameters (1.16) exist for non-trivial solution (1.8), the function (1.1) on the sphere (1.2) takes critical values at points of the form (1.17), and its second variation at these points has the form (1.24) and is the sum of two quadratic forms, one of which depends on two variables and, generally speaking, does not have a diagonal form, while the second depends on the remaining independent variables and has a diagonal form.

*Note*. The index of the quadratic form (24) changes for the same values of the parameters $p$ for which either the determinant of the quadratic form (1.25) changes sign (in this case the functions $\varphi_{\alpha\beta\gamma}(\mathbf{p})$ and $\psi_{\alpha\beta\gamma}(\mathbf{p})$ lose their uniqueness and no solutions appear which differ in principle from (1.17)), or one of the coefficients of the quadratic form (1.26) changes sign. In the latter case, solutions of the following form bifurcate from solutions (1.17)

$$x_\alpha = \pm\cos\varphi\cos\psi\cos\chi, \quad x_\beta = \pm\sin\varphi\cos\psi\cos\chi$$

$$x_\gamma = \pm\sin\psi\cos\chi, \quad x_\delta = \pm\sin\chi, \quad x_j = 0 \quad (j \neq \alpha, \beta, \gamma, \delta)$$

where the subscript $\delta$ corresponds to the Poincaré coefficient $c_\delta^{(\alpha\beta\gamma)}$, which changes sign, etc.

To conclude this part of the paper we note that to find the non-trivial solutions (1.8) it is sufficient to solve one non-linear algebraic equation (1.10), to find the solutions (1.17) it is sufficient to solve a system of two such equations (1.19) and (1.20), etc. Even if it is impossible to obtain solutions of these equations analytically, an analysis or numerical solution of them is much simpler than an analysis or numerical solution of the initial system of equations (1.3).

**2.** We will now assume that the potential energy depends on $2n$ variables $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$, where

$$V(\mathbf{x};\ \mathbf{y};\ \mathbf{p}) = W(\boldsymbol{\xi};\ \boldsymbol{\eta};\ \mathbf{p})\,(\xi_i = x_i^2,\ \eta_i = y_i^2) \tag{2.1}$$

and the variables $\mathbf{x}$ and $\mathbf{y}$ are related by the equations

$$\begin{aligned} &f(\mathbf{x}) = x_1^2 + \ldots + x_n^2 - 1 = 0, \quad g(\mathbf{x}) = y_1^2 + \ldots + y_n^2 - 1 = 0 \\ &h(\mathbf{x},\ \mathbf{y}) = x_1 y_1 + \ldots + x_n y_n = 0 \end{aligned} \tag{2.2}$$

To find the critical points of the function (2.1) on the set (2.2) we introduce the function $2F = V + \lambda f + \mu g + 2\nu h$, where $\lambda$, $\mu$, $\nu$ are Lagrange undetermined multipliers, and we write the conditions for it to be steady with respect to the variables $\mathbf{x}$, $\mathbf{y}$, $\lambda$, $\mu$, $\nu$

$$\begin{aligned} &\frac{\partial F}{\partial x_i} = \left(\frac{\partial W}{\partial \xi_i} + \lambda\right)x_i + \nu y_i = 0 \\ &\frac{\partial F}{\partial y_i} = \left(\frac{\partial W}{\partial \nu_i} + \mu\right)y_i + \nu x_i = 0, \quad i = 1,\ \ldots,\ n \\ &\frac{\partial F}{\partial \lambda} = \frac{1}{2}f = 0, \quad \frac{\partial F}{\partial \mu} = \frac{1}{2}g = 0, \quad \frac{\partial F}{\partial \nu} = h = 0 \end{aligned} \tag{2.3}$$

System (2.3) obviously has the following trivial solutions

$$\begin{aligned} &x_\alpha = \pm 1, \quad y_\beta = \pm 1 \quad (\alpha \neq \beta) \\ &x_j = 0 \quad (j \neq \alpha), \quad y_k = 0 \quad (k \neq \beta) \end{aligned} \tag{2.4}$$

$$\lambda = \lambda_{\alpha\beta} = -\left[\frac{\partial W}{\partial \xi_\alpha}\right]_{(2.4)}, \quad \mu = \mu_{\alpha\beta} = -\left[\frac{\partial W}{\partial \nu_\beta}\right]_{(2.4)}, \quad \nu = 0 \tag{2.5}$$

To determine the nature of the critical points (2.4) we will calculate (taking (2.5) into account) the second variation $\delta^2 F$ of the function $F$ on the linear manifold $\delta f = 0$, $\delta g = 0$, $\delta h = 0$, which for solutions (2.4) has the form $\delta x_\alpha = 0$, $\delta y_\beta = 0$, $\delta x_\beta + \delta y_\alpha = 0$ (here and henceforth, in view of the symmetry of the problem, all the solutions are taken with the upper sign). Hence

$$2\delta^2 F = \sum_{j \neq \alpha,\beta} [c_j^{(\alpha\beta)}(\delta x_j)^2 + d_j^{(\alpha\beta)}(\delta y_j)^2] + [c_\beta^{(\alpha\beta)} + d_\alpha^{(\alpha\beta)}](\delta z)^2 \tag{2.6}$$

$$\delta z = \delta x_\beta = -\delta y_\alpha$$

$$c_j^{(\alpha\beta)} = \left[\frac{\partial W}{\partial \xi_j} - \frac{\partial W}{\partial \xi_\alpha}\right]_{(2.4)} \quad (j \neq \alpha); \qquad d_k^{(\alpha\beta)} = \left[\frac{\partial W}{\partial \eta_k} - \frac{\partial W}{\partial \eta_\beta}\right]_{(2.4)} \quad (k \neq \beta)$$

Thus the following assertion holds.

*Assertion* 2.1. The function (2.1) of the set (2.2) always takes critical values at the points (2.4), and its second variation at these points always has the form of the sum of the squares (2.6).

We will consider one of the trivial solutions (2.4) and we will assume that for certain values of the parameters of the Poincaré coefficients of the second variation (2.6) changes sign, while the remaining coefficients do not vanish for these values of the parameters. We will initially assume that $c_\gamma^{(\alpha\beta)}$ or $d_\gamma^{(\alpha\beta)}$ ($\gamma \neq \alpha, \beta$) changes sign. Then, in exactly the same way as in the previous section, it can be shown that system (2.3) has non-trivial solutions of the form

$$x_\alpha = \pm\cos\varphi, \quad x_\gamma = \pm\sin\varphi$$
$$y_\beta = \pm 1, \quad x_j = y_k = 0 \ (j \neq \alpha, \ \gamma; \ k \neq \beta) \tag{2.7}$$

(if $c_\gamma^{(\alpha\beta)}$ changes sign) or

$$x_\alpha = \pm 1, \quad y_\beta = \pm\cos\varphi, \quad y_\gamma = \pm\sin\varphi$$
$$x_j = y_k = 0 \ (j \neq \alpha; \ k \neq \beta, \ \gamma) \tag{2.8}$$

(if $d_\gamma^{(\alpha\beta)}$ changes sign). Here

$$\varphi = \pm\varphi_{\alpha\gamma}(\mathbf{p}) : \Phi_{\alpha\gamma}(\varphi, \ \mathbf{p}) = 0 \tag{2.9}$$

$$\Phi_{\alpha\beta\gamma} = \frac{\partial W_{\alpha\beta\gamma}}{\partial \zeta_\gamma} - \frac{\partial W_{\alpha\beta\gamma}}{\partial \zeta_\alpha}; \quad W_{\alpha\beta\gamma} = W|_{(2.7) \text{ or } (2.8)}; \quad \zeta = \xi \text{ or } \eta$$

$$\lambda = \lambda_{\alpha\beta} = -\left[\frac{\partial W}{\partial \xi_\alpha}\right]_{(2.7) \text{ or } (2.8)}, \quad \mu = \mu_{\alpha\beta\gamma} = -\left[\frac{\partial W}{\partial \eta_\beta}\right]_{(2.7) \text{ or } (2.8)}, \quad \nu = 0 \tag{2.10}$$

The second variation $\delta^2 F$ is calculated on the linear manifold $\delta f = 0$, $\delta g = 0$, $\delta h = 0$ which, for solution (2.7), has the form

$$\cos\varphi\delta x_\alpha + \sin\varphi\delta x_\gamma = 0, \quad \delta y_\beta = 0, \quad \cos\varphi\delta y_\alpha + \sin\varphi\delta y_\gamma + \delta x_\beta = 0 \tag{2.11}$$

while for solution (2.8) it has the form

$$\delta x_\alpha = 0, \quad \cos\varphi\delta y_\beta + \sin\varphi\delta y_\gamma = 0, \quad \cos\varphi\delta x_\beta + \sin\varphi\delta x_\gamma + \delta y_\alpha = 0 \tag{2.12}$$

Hence

$$[2\delta^2 F]_{(2.7)} = [2\delta^2 F]^{(j\neq\alpha,\beta,\gamma)} + [2\delta^2 F]^{(\gamma;\alpha,\gamma)} \tag{2.13}$$

$$[2\delta^2 F]_{(2.8)} = [2\delta^2 F]^{(j\neq\alpha,\beta,\gamma)} + [2\delta^2 F]^{(\beta,\gamma;\gamma)} \tag{2.14}$$

$$[2\delta^2 F]^{(j\neq\alpha,\beta,\gamma)} = \sum_{j\neq\alpha,\beta,\gamma} [c_j^{(\alpha\beta\gamma)}(\delta x_j)^2 + d_j^{(\alpha\beta\gamma)}(\delta y_j)^2]$$

$$[2\delta^2 F]^{(\gamma;\alpha,\gamma)} = c_\gamma^{\alpha\beta\gamma}(\delta x_\beta)^2 + b_{\alpha\alpha}(\delta y_\alpha)^2 + 2b_{\alpha\gamma}(\delta y_\alpha)(\delta y_\gamma) + b_{\gamma\gamma}(\delta y_\gamma)^2$$

$$[2\delta^2 F]^{(\beta,\gamma;\gamma)} = a_{\beta\beta}(\delta x_\beta)^2 + 2a_{\beta\gamma}(\delta x_\beta)(\delta x_\gamma) + a_{\gamma\gamma}(\delta x_\gamma)^2 + d_\gamma^{\alpha\beta\gamma}(\delta y_\beta)^2$$

(explicit expressions for the coefficients of the quadratic forms (2.13) and (2.14) are quite lengthy and are therefore not given here).

Thus the following assertion holds.

*Assertion* 2.2. If bifurcation values of the parameters $c_\gamma^{(\alpha\beta)}(\mathbf{p}) = 0$ or $d_\gamma^{(\alpha\beta)}(\mathbf{p}) = 0$ $(\gamma\neq\alpha, \beta)$ exist for the trivial solution (2.4), the function (2.1) on the set (2.2) takes critical values at points of the form (2.7) or (2.8) respectively, and its second variation at these points has the form (2.13) or (2.14).

*Note*. Solutions (2.7) and (2.8) are to some extent analogous to solutions (1.8), considered in the previous section. The coefficients $c_\gamma^{(\alpha\beta\gamma)}(\mathbf{p})$ or $d_\gamma^{(\alpha\beta\gamma)}(\mathbf{p})$ of quadratic form (2.13) or (2.14) vanish when $c_\gamma^{(\alpha\beta)}(\mathbf{p}) = 0$ or $d_\gamma^{(\alpha\beta)}(\mathbf{p}) = 0$. If these coefficients change sign for some other values of the parameters, the functions $\varphi_{\alpha\beta\gamma}(\mathbf{p})$ lose their uniqueness. If some other Poincaré coefficients $c_j^{(\alpha\beta\gamma)}(\mathbf{p})$ or $d_j^{(\alpha\beta\gamma)}(\mathbf{p})$ $(j\neq \alpha, \beta, \gamma)$ of quadratic forms (2.13) and (2.14) or expressions $b_{\alpha\alpha}b_{\gamma\gamma} - b_{\alpha\gamma}^2$ and $a_{\beta\beta}a_{\gamma\gamma} - a_{\alpha\gamma}^2$ corresponding to these forms change their signs, non-trivial solutions of the second level, etc. will bifurcate from solutions (2.7) and (2.8).

We will now consider the case when the Poincaré coefficient $c_\beta^{(\alpha\beta)} + d_\alpha^{(\alpha\beta)}$ of the second variation (2.6), calculated for the trivial solution (2.4), changes sign on a certain set

$$\mathbf{P}_{\alpha\beta} = \{\mathbf{p}\in\mathbf{R}^m : c_\beta^{(\alpha\beta)}(\mathbf{p}) + d_\alpha^{(\alpha\beta)}(\mathbf{p}) = 0\} \tag{2.15}$$

Here, as previously, we will assume that the remaining coefficients of the second variation (2.6) do not vanish on the sets (2.15).

In this case the following solutions bifurcate from solutions (2.4)

$$x_\alpha = \pm\cos\varphi,\quad x_\beta = \pm\sin\varphi$$

$$y_\alpha = \mp\sin\varphi,\quad y_\beta = \pm\cos\varphi,\quad x_j = y_j = 0\ (j\neq\alpha,\ \beta) \tag{2.16}$$

In fact, substituting (2.16) into system (2.3) we obtain (as previously we take the upper signs in relations (2.16))

$$\left(\frac{\partial W_{\alpha\beta}}{\partial\xi_\alpha} + \lambda\right)\cos\varphi - \nu\sin\varphi = 0,\quad \left(\frac{\partial W_{\alpha\beta}}{\partial\xi_\beta} + \lambda\right)\sin\varphi + \nu\cos\varphi = 0 \tag{2.17}$$

$$\left(\frac{\partial W_{\alpha\beta}}{\partial\nu_\alpha} + \mu\right)\sin\varphi + \nu\cos\varphi = 0,\quad \left(\frac{\partial W_{\alpha\beta}}{\partial\nu_\beta} + \mu\right)\cos\varphi + \nu\sin\varphi = 0$$

where $W_{\alpha\beta} = W|_{(2.16)}$, while the remaining equations of system (2.3) are satisfied identically with respect to $\varphi$, $\lambda$, $\mu$ and $\nu$. Eliminating the undetermined multipliers from (2.17), we obtain

$$\Phi_{\alpha\beta}(\varphi;\ \mathbf{p}) \equiv \frac{\partial W_{\alpha\beta}}{\partial\xi_\beta}(1-\sin\varphi) - \frac{\partial W_{\alpha\beta}}{\partial\xi_\alpha}\cos\varphi + \frac{\partial W_{\alpha\beta}}{\partial\eta_\alpha}(1-\sin\varphi) - \frac{\partial W_{\alpha\beta}}{\partial\eta_\beta}\cos\varphi = 0 \tag{2.18}$$

By the above assumptions $\Phi_{\alpha\beta}(0; \mathbf{p}) \equiv c_\beta^{(\alpha\beta)}(\mathbf{p}) + d_\alpha^{(\alpha\beta)}(\mathbf{p})$ changes sign on the set (2.15) and hence the analysis of Eq. (2.18) is analogous to the analysis of Eq. (1.10) carried out earlier. Hence, Eq. (2.18) always has a solution of the form

$$\varphi = \pm\varphi_{\alpha\beta}(\mathbf{p}) \tag{2.19}$$

for values of the parameters lying in the neighbourhood of the set (2.15).

Moreover, taking (2.19) into account it follows from (2.17) that

$$\lambda = \lambda_{\alpha\beta} = -\left[\frac{\partial W}{\partial \xi_\alpha}\cos\varphi + \frac{\partial W}{\partial \xi_\beta}\sin\varphi\right]_{(2.16)}$$

$$\mu = \mu_{\alpha\beta} = -\left[\frac{\partial W}{\partial \eta_\beta}\cos\varphi + \frac{\partial W}{\partial \eta_\alpha}\sin\varphi\right]_{(2.16)}$$

$$\nu = \nu_{\alpha\beta} = \left[\frac{\partial W}{\partial \xi_\alpha}\sin\varphi - \frac{\partial W}{\partial \xi_\beta}\cos\varphi\right]_{(2.16)} \equiv \left[\frac{\partial W}{\partial \eta_\alpha}\cos\varphi - \frac{\partial W}{\partial \eta_\beta}\sin\varphi\right]_{(2.16)}$$

The second variation $\delta^2 F$ is calculated on the linear manifold $\delta f = 0$, $\delta g = 0$, $\delta h = 0$, which, for solution (2.16), has the form

$$\cos\varphi\delta x_\alpha + \sin\varphi\delta x_\beta = 0, \quad -\sin\varphi\delta y_\alpha + \cos\varphi\delta y_\beta = 0$$

$$\cos\varphi\delta y_\alpha - \sin\varphi\delta x_\alpha + \sin\varphi\delta y_\beta + \cos\varphi\delta x_\beta = 0$$

Hence

$$2\delta^2 F = [2\delta^2 F]^{(j\neq\alpha,\beta)} + e(\delta z)^2 \quad (\delta z = -\delta x_\beta = \delta y_\alpha) \tag{2.20}$$

$$[2\delta^2 F]^{(j\neq\alpha,\beta)} = \sum_{j\neq\alpha,\beta} [c_j^{(\alpha\beta)}(\delta x_j)^2 + d_j^{(\alpha\beta)}(\delta y_j)^2 + 2e_j^{(\alpha\beta)}(\delta x_j)(\delta y_j)]$$

The explicit expressions for the coefficients $c_j$, $d_j$ and $e_j$, and, particularly, $e$, are fairly lengthy and are therefore not given here.

Hence, the following assertion holds.

*Assertion* 2.3. If bifurcation values of the parameters (2.15) exist for the non-trivial solution (2.4), the function (2.1) on the set (2.2) takes critical values at points of the form (2.16), and its second variation at these points has the form (2.20).

*Note.* The coefficient $e$ of the second variation (2.20) vanishes on the set (2.15). If this coefficient changes sign for certain other values of the parameters, the functions (2.19) lose their uniqueness. If any expression $c_\gamma d_\gamma - e_\gamma^2$ changes sign, the non-trivial solutions of the second level, etc. bifurcate from solutions (2.16).

In conclusion we will consider the problem of the relative equilibria of a rigid body in a circular orbit in a central gravitational field. We will assume that the mass distribution of the body is symmetrical about planes passing through any pair of its principal central axes of inertia. Then (apart from a constant; see, for example, [2]) the changed potential energy of the body has the form

$$V = \mu\frac{M}{2R^3}[3(J_1\gamma_1^2 + J_2\gamma_2^2 + J_3\gamma_3^2) - (J_1\beta_1^2 + J_2\beta_2^2 + J_3\beta_3^2) + \varepsilon^2 F(\gamma_1^2, \gamma_2^2, \gamma_3^2, \varepsilon^2)] \tag{2.21}$$

Here $\mu$ is the gravitational constant, $M$ is the mass of the attracting centre, $R$ is the radius of the orbit of the centre of mass of the body, $J_1$, $J_2$ and $J_3$ are the principal central moments of inertia, $\varepsilon$ is the ratio of the characteristic dimension of the body to the radius of the orbit, $\gamma_1$, $\gamma_2$, $\gamma_3$ are the direction cosines of the radius vector of the centre of mass of the body with respect to the attracting centre, $\beta_1, \beta_2, \beta_3$ are the direction cosines of the normal to the plane of the orbit and $F$ is a certain function, whose explicit form depends on the mass distribution of the body. It is obvious that the variables $\gamma$ and $\beta$ are constrained by the relations

$$\gamma_1^2+\gamma_2^2+\gamma_3^2-1=0, \quad \beta_1^2+\beta_2^2+\beta_3^2-1=0, \quad \gamma_1\beta_1+\gamma_2\beta_2+\gamma_3\beta_3=0 \tag{2.22}$$

By what was said above, the function (2.21) under conditions (2.22) takes steady values at the points

$$\gamma_i=\pm 1, \quad \beta_j=\pm 1 \quad (i\neq j) \tag{2.23}$$
$$\gamma_j=\gamma_k=\beta_i=\beta_k=0 \quad (i, \ j, \ k=1, \ 2, \ 3)$$

corresponding to trivial equilibrium orientations of the bodies. Here the $i$th axis of the body is directed along the radius vector, the $j$th axis is directed along the normal to the plane of the orbit, while the $k$th axis is directed along the tangent to the orbit.

The second variation of the function (2.21) on a linear manifold $\delta\gamma_i=0$, $\delta\beta_j=0$, $\delta\beta_i\pm\delta\gamma_j=0$ has the form ($F_s$ denotes $\partial F/\partial\gamma_s^2$)

$$2\delta^2 V=c_k(\delta\gamma_k)^2+d_k(\delta\beta_k)^2+(c_j+d_i)(\delta\gamma_j)^2 \tag{2.24}$$
$$c_s=3(J_s+J_i)+\varepsilon^2(F_s-F_i)_{(2.23)} \quad (s=j, \ k), \quad d_r=J_j-J_r \quad (r=i, \ k)$$

In the case of a body with a triaxial ellipsoid of inertia $d_r\neq 0$. Consequently, there are no non-trivial equilibrium orientations of the body, for which it turns around the radius vector. If the value of $J_k$ is close to $J_i$, then $c_k$ may vanish and change sign for a certain value of $\varepsilon$, close to zero. In this case non-trivial equilibrium orientations are produced of the form

$$\gamma_i=\pm\cos\varphi, \quad \gamma_k=\pm\sin\varphi, \quad \beta_j=\pm 1, \quad \gamma_j=\beta_i=\beta_k=0 \tag{2.25}$$

for which the body is turned by an angle $\varphi$ around the normal to the plane of the orbit. If the value of $J_j$ is close to $J_i$, then $c_j+d_i$ may vanish and change sign for a certain value of $\varepsilon$, close to zero. In this case the following non-trivial equilibrium orientations are produced

$$\gamma_i=\pm\cos\varphi, \quad \gamma_j=\pm\sin\varphi$$
$$\beta_j=\pm\cos\varphi, \quad \beta_i=\mp\sin\varphi, \quad \gamma_k=\beta_k=0 \tag{2.26}$$

for which the body is turned by an angle $\varphi$ around the tangent to the orbit. Further analysis of the problem depends very much on the explicit expression for the function $F$. Some special cases of problems of relative equilibria and steady motions of a body were considered previously in [2–5].

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